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THE STEADY MOTION OF A CRACK IN AN ELASTIC STRIP (HOMOGENEOUS PROBLEMS)†

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The steady dynamic problem of the motion of a central semi-infinite crack $x \in (-\infty, 0), y = 0$ at a sub-Rayleigh velocity in an elastic strip $x \in (-\infty, +\infty), y \in (-h, h)$ is considered. The amplitudes of the propagating waves (the homogeneous solutions) are specified at $x = -\infty$. The related problem of the delamination of two strips $x \in (-\infty, +\infty), y \in (-h, 0)$ and $y \in (0, h)$ which are not bonded together but closely adjoin one another when $x \geq 0, y = 0$ is solved in quadratures together with the problem of the cleavage of the strip. The stress intensity factors at the crack tip are found. Cases of the partial flux of the deformation energy along x from minus infinity to plus infinity by passing the crack are investigated. The continuous piecewise-homogeneous solutions of elastostatic and steady problems are compared. In particular, it is shown that steady solutions, which when $c \to 0$, become the well-known elastostatic solutions of problems in which bending moments and shearing forces are applied to the cleaved parts when $x = -\infty$, do not exist. © 2004 Elsevier Ltd. All rights reserved.

In many problems of the quasi-steady crack growth, the main characteristic of the crack stability, the stress intensity factor, is found without solving the problem. Rice [1] has developed the corresponding technique. It is based on the principle of the conservation of the flux of energy G passing through an elastic body from infinity to the crack tip and on Irwin's formula [2], which relates the stress intensity factor to the quantity G. Problems of this kind of different complexity have been considered in [3]. If the limiting sink of energy G_C is used as the criterion for the crack growth rather than the critical stress intensity factor for a given material K_C, then, in order to estimate the crack state, the increment of the energy G is calculated using elementary mechanical considerations corresponding to the conditions at infinity and, also, the criterion for the crack instability $G \geq G_C$.

The results of the investigation carried out below enable us to note that, in steady problems of elasticity for waveguides, the above mentioned technique for investigating a crack does not work as the energy and inertia of an elastic body at infinity are unboundedly large and it is impossible to guess in advance what the flux will be. However, as follows from this paper, a still more important obstacle to the use of the Rice method in steady problems is the fact that the condition for the steady crack growth $G = G_C$ is not satisfied. It is found that, in a number of cases, the power of the energy flux generated by the propagating waves E when $x = -\infty$ is several times greater than the limiting value of E_C, since part of it, on by-passing the crack tip $x \in (-\infty, 0)$, departs to $+\infty$. The propagating waves, which have their own frequency after passing through the section $x = 0$, serve as carriers of this flux.

1. FORMULATION OF THE PROBLEM

Together with an elastic plane, an elastic strip, which is weakened by a semi-infinite cut, serves as the simplest and most informative model of the crack state in a solid. It takes account of the boundedness of the real domains in one direction, and describes the properties of the propagating waves and their contribution to the influx (or outflux) of additional energy to the crack tip.

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Suppose a strip $x_1 \in (-\infty, +\infty), y_1 \in (-h, h)$ with elastic characteristics λ and μ and density ρ, moves at a constant sub-Rayleigh velocity c relative to the xOy plane in a direction opposite to the Ox axis such that $x = x_1 - ct, y = y_1$, where t is the time.

The strip is cleaved by a semi-infinite central crack $x < 0, y = 0$, the tip of which moves in it with a velocity c in the direction of O_1x_1 and remains in place in the coordinates xOy.

Because of the mirror symmetry of the cleaved strip $x \in (-\infty, +\infty), y \in (-h, h)$, the initial problem decomposes into a symmetric problem (problem A) and a skew-symmetric problem (problem B). These problems can be formulated in the form of two problems for the upper strip $x \in (-\infty, +\infty), y \in (0, h)$ with mixed boundary conditions

for both problems

$$\sigma_y(x, h) = \tau_{xy}(x, h) = 0, \quad x \in (-\infty, +\infty) \tag{1.1}$$

for problem A

$$\begin{gathered}\sigma_y(x, 0) = -P_1\delta(x+\xi), \quad x \in (-\infty, 0); \quad v(x, 0) = 0, \quad x \in (0, +\infty) \\ \tau_{xy}(x, 0) = 0, \quad x \in (-\infty, +\infty)\end{gathered} \tag{1.2}$$

for problem B

$$\begin{gathered}\tau_{xy}(x, 0) = P_2\delta(x+\xi), \quad x \in (-\infty, 0); \quad u(x, 0) = 0, \quad x \in (0, +\infty) \\ \sigma_y(x, 0) = 0, \quad x \in (-\infty, +\infty)\end{gathered} \tag{1.3}$$

where $P_1 \geq 0$ and $P_2 \geq 0$ are the magnitudes of the compressive concentrated shear forces, normal to and in the opposite direction to the Ox axis, which are applied to the crack surfaces at the points $x = -\xi, y = 0$ and $\delta(x)$ is the Dirac δ-function. Moreover, certain wave conditions when $x = -\infty$ and conditions that there is no contact between the crack surfaces $v(x, 0) \geq 0, x < 0$ are specified in problem A. In considering the cleavage problems A and B, the condition that the elastic strain energy must be bounded in the neighbourhood of the crack tip must be satisfied. In the delamination problem A, the boundedness of the compressive normal stresses $\sigma_y(0, 0) \leq 0$ and the absence of tensile stresses $\sigma_y(x, 0) > 0$ when $x > 0$ are necessary.

2. THE GENERAL SOLUTION

The solution is constructed using Papkovich–Neuber functions in Laplace integrals [4]

$$u(x, y) = \frac{1}{2\pi i}\int_L \left[p\Phi(p, y) + \frac{2}{(a^2-b^2)p}\Psi'(p, y)\right]e^{px}dp \tag{2.1}$$

$$v(x, y) = \frac{1}{2\pi i}\int_L \left[\Phi'(p, y) - \frac{2}{a^2-b^2}\Psi(p, y)\right]e^{px}dp \tag{2.2}$$

$$\Phi(p, y) = A\cos apy + B\sin apy, \quad \Psi(p, y) = C\cos bpy + D\sin bpy \tag{2.3}$$

$$a = \sqrt{1-c^2c_1^{-2}}, \quad b = \sqrt{1-c^2c_2^{-2}}, \quad c_1^2 = (\lambda+2\mu)\rho^{-1}, \quad c_2^2 = \mu\rho^{-1} \tag{2.4}$$

where c_1 and c_2 are the velocities of the compression and shear waves.

Using the Cauchy formulae and substituting the functions (2.1)–(2.4) into the main equations (1.1)–(1.3), we obtain the general formulae for problems A and B

$$u_k(x, y) = \frac{1}{2\pi i}\int_L G(p)U_k(p, y)e^{px}dp, \quad k = 1, 2, \ldots, 5 \tag{2.5}$$

$$U_1(p, y) = p(Ac_a + Bs_a) + \frac{2b}{a^2-b^2}(-Cs_b + Dc_b) \tag{2.6}$$

$$U_2(p,y) = ap(-As_a + Bc_a) - \frac{2}{a^2-b^2}(Cc_b + Ds_b) \tag{2.7}$$

$$U_3(p,y) = 2\mu p\left[ap(-As_a + Bc_a) - \frac{1+b^2}{a^2-b^2}(Cc_b + Ds_b)\right] \tag{2.8}$$

$$U_4(p,y) = p\left\{p[\lambda(1-a^2)+2\mu](Ac_a + Bs_a) + \frac{4b\mu}{a^2-b^2}(-Cs_b + Dc_b)\right\} \tag{2.9}$$

$$U_5(p,y) = -\mu p\left[p(1+b^2)(Ac_a + Bs_a) + \frac{4b}{a^2-b^2}(-Cs_b + Dc_b)\right] \tag{2.10}$$

where $U_k(p,y)$ are the Laplace transforms of the displacements $u(x,y) = u_1(x,y)$, $\upsilon(x,y) = u_2(x,y)$ and the stresses $\tau_{xy}(x,y) = u_3(x,y)$, $\sigma_x(x,y) = u_4(x,y)$, $\sigma_y(x,y) = u_5(x,y)$, $G(p)$ is an arbitrary function and $c_r = \cos rpy$, $s_r = \sin rpy$.

In the symmetric problem A

$$A = (1+b^2)(4ab - c_+)p^{-1}, \quad B = (1+b^2)d_- p^{-1} \tag{2.11}$$

$$C = a(a^2-b^2)d_-, \quad D = a(a^2-b^2)[c_- + (1+b^2)^2] \tag{2.12}$$

In the skew-symmetric problem B

$$A = -4bd_+p^{-1}, \quad B = 4b[c_- + (1+b^2)^2]p^{-1} \tag{2.13}$$

$$C = (a^2-b^2)(1+b^2)(c_+ - 4ab), \quad D = (a^2-b^2)(1+b^2)d_+ \tag{2.14}$$

Here,

$$\alpha_\pm = [4ab \pm (1+b^2)^2]/2$$

$$c_\pm = \alpha_-\cos(a+b)ph \pm \alpha_+\cos(a-b)ph$$

$$d_\pm = \pm\alpha_-\sin(a+b)ph - \alpha_+\sin(a-b)ph$$

3. HOMOGENEOUS PROBLEMS

Putting $P_1 = P_2 = 0$ in the mixed conditions (1.2) and (1.3) and substituting the functions (2.10), (2.7), (2.11), (2.12) and (2.8), (2.6), (2.13), (2.14), respectively, instead of their transforms, in the case of homogeneous problems we obtain the following equalities

for problem A

$$\sigma^+(p) = G(p)N_1(p), \quad \upsilon^-(p) = G(p)N_2(p), \quad p \in L \tag{3.1}$$

$$N_1(p) = 4\mu p\left(\alpha_-^2\sin^2\frac{a+b}{2}ph - \alpha_+^2\sin^2\frac{a-b}{2}ph\right), \quad N_2(p) = -a(1-b^2)d_- \tag{3.2}$$

for problem B

$$\tau^+(p) = G(p)N_3(p), \quad u^-(p) = G(p)N_4(p), \quad p \in L \tag{3.3}$$

$$N_3(p) = -2N_1(p), \quad N_4(p) = -2b(1-b^2)d_+ \tag{3.4}$$

The plus and minus superscripts denote the analyticity of the transforms of the functions $\sigma_y(x,0)$, $\tau_{xy}(x,0)$ and $\upsilon(x,0)$, $u(x,0)$ on the right-hand side $\mathrm{Re}\,p > 0$ and the left-hand side $\mathrm{Re}\,p < 0$ of the complex plane p respectively. On eliminating the function $G(p)$ in (3.1) and (3.3), we obtain the Wiener–Hopf equations:

for problem A

$$\sigma^+(p) = K(p)v^-(p), \quad K(p) = N_1(p)/N_2(p), \quad p \in L \tag{3.5}$$

for problem B

$$\tau^+(p) = K(p)u^-(p), \quad K(p) = N_3(p)/N_4(p), \quad p \in L \tag{3.6}$$

In this case, it is those piecewise-homogeneous solutions of the problems being considered which describe elastic wave that propagate without attenuation to or from infinity which are of interest. Obviously, they are only generated by the zeros of the function N_k (3.2), (3.4), lying on the imaginary axis. We shall now study these zeros.

Since the quantities a, b, $a - b$ and $\alpha_\pm$ are positive, $\sin(a \pm b)ph > 0$ on the imaginary axis $p = i\beta$ when $\beta > 0$ and, consequently, the function $N_2(p)$ has a simple zero $p = 0$ and does not have other pure imaginary zeros when $\beta \in (-\infty, +\infty)$.

When $p = i\beta$, $\beta > 0$, the function $N_4(p)$ obviously has only the same zeros as the function $f(\beta) - \alpha_-\alpha_+^{-1}$, where

$$f(\beta) = \operatorname{sh}(a-b)\beta h[\operatorname{sh}(a+b)\beta h]^{-1}$$

Since

$$(a\operatorname{sh}2b\beta h - b\operatorname{sh}2a\beta h)'_\beta = -4abh\operatorname{sh}(a+b)\beta h\operatorname{sh}(a-b)\beta h < 0, \quad \beta \in (0, \infty)$$

$$\lim_{\beta \to 0}(a\operatorname{sh}2b\beta h - b\operatorname{sh}2a\beta h) = 0$$

then

$$f'(\beta) = h(a\operatorname{sh}2b\beta h - b\operatorname{sh}2a\beta h)[\operatorname{sh}(a+b)\beta h]^{-2} < 0, \quad \beta \in (0, \infty)$$

Consequently, the functions $f(\beta)$ and $f(\beta) - \alpha_-\alpha_+^{-1}$ decay monotonically with respect to β in the interval $(0, \infty)$. According to the equalities

$$\lim_{\beta \to 0} f(\beta) = (a-b)(a+b)^{-1}, \quad \lim_{\beta \to \infty} f(\beta) = 0$$

the last function has opposite signs at the ends of this interval:

$$f(0) - \frac{\alpha_-}{\alpha_+} = \frac{a(1-b^2)^2}{\alpha_+(a+b)} > 0, \quad f(\infty) - \frac{\alpha_-}{\alpha_+} = -\frac{\alpha_-}{\alpha_+} < 0$$

Hence, in the interval $(0, \infty)$ the functions $f(\beta) - \alpha_-\alpha_+^{-1}$ and $N_4(i\beta)$ have the unique simple zero $\beta = \beta_0$. Moreover, the function $N_4(p)$ on the imaginary axis has the simple zeros $p = 0$ and, by virtue of its oddness, $p = -i\beta_0$.

The function $N_1(p)$ can be represented in the form of a product of the functions investigated

$$N_1(p) = -\frac{2\mu p}{ab(1-b^2)^2} N_2\left(\frac{p}{2}\right) N_4\left(\frac{p}{2}\right)$$

It follows from this that, in the case of the same parameters in all of the $N_k(p)$, the functions $N_1(p)$ and $N_3(p)$ have two simple zeros $p = \pm 2i\beta_0$ on the imaginary axis and a triple zero $p = 0$.

The functions $N_k(p)$ also have a denumerable set of complex and real zeros $p = p_n(n = \pm 1, \pm 2, \ldots)$ which are arranged symmetrically about the axes $\operatorname{Re} p = 0$ and $\operatorname{Im} p = 0$. On the basis of well-known theory [5], it can be shown that their asymptotic form is

$$p_n = \frac{\pi n}{(a+b)h} + r_n, \quad n = \pm 1, \pm 2, \ldots$$

where n is the number of zeros in the half-plane $\operatorname{Re} p > 0$ in order of non-decreasing modulus p_r and r_n are complex numbers which are bounded with respect to all n and with respect to their modulus in the general case.

We will seek the solution of problem (3.5) in the form of the product of solutions of two Riemann problems [6]:

$$\sigma_j^+(p) = K_j(p)v_j^-(p), \quad j = 1, 2, \quad p \in L \tag{3.7}$$

$$\sigma^+(p) = \sigma_1^+(p)\sigma_2^+(p), \quad v^-(p) = v_1^-(p)v_2^-(p)$$

$$K_1(p) = Hp\operatorname{tg}\pi p\operatorname{tg}\pi(p-2i\beta_0)\operatorname{tg}\pi(p+2i\beta_0), \quad K_2(p) = K(p)/K_1(p) \tag{3.8}$$

$$H = -2\mu\alpha_-[a(1-b^2)]^{-1}$$

Here, the coefficient $K_1(p)$ is taken such that the function $K_2(p)$ satisfies Hölder's condition on the whole of the imaginary axis, is even, and is strictly positive and, consequently, has an index equal to zero $K_2(i\beta) \to 1$ when $\beta \to \pm\infty$. The factorization of $K_1(p)$ depends on the shape of the contours L which we shall choose such that they pass along the imaginary axis and bend along the arcs of semicircles of small radius to the left and to the right of the points $p = 0$ and $p = \pm 2i\beta_0$. In all, there are 2^3 such contours L_q ($q = 1, 2, \ldots$), the first three of which are shown in Fig. 1.

We now show that the solutions in L_1, L_2 include all the piecewise-homogeneous solutions which satisfy the boundary and energy conditions which have been imposed and that the solutions in L_3 are linear combinations of the solutions in L_1 and L_2. When $q > 3$, solutions do not exist in the class of specified functions.

We will now put $g_{rq}(p) = g_r(p)$ when $p \in L_q$. On expanding the tangents in products of gamma-functions, from relations (3.7) and (3.8) when $q = 1, 2$ we obtain

$$\sigma_{1q}^+(p) = H[p-2(-1)^q i\beta_0]Q(p), \quad v_{1q}^-(p) = \{p^2[p+2(-1)^q i\beta_0]Q(-p)\}^{-1}$$

$$Q(p) = \frac{\Gamma(1/2+p)\Gamma(1/2+p-2i\beta_0)\Gamma(1/2+p+2i\beta_0)}{\Gamma(1+p)\Gamma(1+p-2i\beta_0)\Gamma(1+p+2i\beta_0)}$$

It follows from this that

$$\sigma_1^+(p) \sim Hp^{-1/2}, \quad v_1^-(p) \sim p^{-3/2}, \quad p \to \infty; \quad \sigma_1^+(0) = (-1)^{q+1} iH\sqrt{\pi}\operatorname{th}2\pi\beta_0 \tag{3.9}$$

Similarly, when $p \in L_3$

$$\sigma_1^+(p) = HQ(p), \quad v_1^-(p) = [p^2(p^2+4\beta_0^2)Q(-p)]^{-1}$$

$$\sigma_1^+(p) \sim Hp^{-3/2}, \quad v_1^-(p) \sim p^{-5/2}, \quad p \to \infty; \quad \sigma_1^+(0) = H\sqrt{\pi}\operatorname{th}2\pi\beta_0(2\beta_0)^{-1}$$

On the contour which passes around the left of the point $p = 0$ and around the right of the points $p = \pm 2i\beta_0$, we have

$$\sigma_1^+(p) = H(p^2+4\beta_0^2)Q(p); \quad \sigma_1^+(p) \sim Hp^{1/2}, \quad v_1^-(p) \sim p^{-1/2}, \quad p \to \infty \tag{3.10}$$

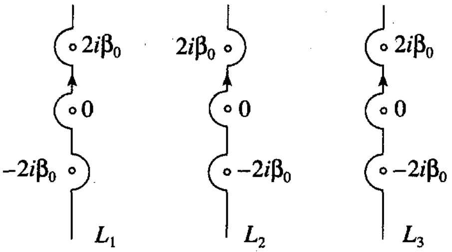

Fig. 1

On the remaining four contours it is obvious that

$$\sigma_1^+(p) \sim O(p^\gamma), \quad \gamma > 0, \quad p \to \infty \tag{3.11}$$

On using Sokhotskii's formulae and evaluating the integrals along the arcs of the semicircles, which are equal to the half residues of the corresponding functions, it is easy to show that the contours L_q when $j = 2$ and for all q can be moved onto the imaginary axis. Hence, the canonical solution σ_{20}^+ of problem (3.7), (3.8) when $j = 2$ is independent of q and, when the properties of the function $K_2(p)$ are taken into account, is given by the Gakhov formulae [6]

$$\sigma_{20}^+(p) = \exp\left\{-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln K_2(t)}{t-p}dt\right\}, \quad \lim_{p\to\infty}\sigma_{20}^+(p) = 1, \quad \operatorname{Re} p > 0 \tag{3.12}$$

$$\sigma_{20}^+(i\beta) = \sqrt{K_2(i\beta)}\exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln K_2(t)}{t-i\beta}dt\right\} \tag{3.13}$$

By virtue of the oddness of the integrand in (3.13) when $\beta = 0$

$$\sigma_{20}^+(0) = \sqrt{K_2(0)} = \frac{(1-b^2)\sqrt{ah}}{\operatorname{th}2\pi\beta_0\sqrt{2\pi\alpha_-}} > 0$$

We will now find the general solution $\sigma^+(p)$ of problems A when $q = 1, 2, 3$. According to relations (3.5), the equalities

$$\frac{\sigma^+(p)}{\sigma_0^+(p)} = \frac{\upsilon^-(p)}{\upsilon_0^-(p)}, \quad \sigma_0^+(p) = \sigma_1^+(p)\sigma_{20}^+(p), \quad \upsilon_0^-(p) = \upsilon_1^-(p)\upsilon_{20}^-(p) \tag{3.14}$$

are satisfied on a contour L_q.

For the cleavage problem in the class of solutions with a non-zero, bounded elastic strain energy in the neighbourhood of the crack tip, we have

$$\sigma_y(x, 0) \sim K_I/\sqrt{2\pi x}, \quad x \to 0$$

where K_1 is the stress intensity factor. Using a theorem of the Abel type, it follows from this that

$$\sigma^+(p) \sim K_I/\sqrt{2p}, \quad p \to \infty \tag{3.15}$$

According to formulae (3.9) and (3.12) when $q = 1, 2$

$$\sigma_0^+(p) \sim Hp^{-1/2}, \quad p \to \infty \tag{3.16}$$

Since the ratio of these functions (3.14) is equal to a constant, by the generalized Liouville theorem the general solution has the form

$$\sigma^+(p) = M\sigma_0^+(p), \quad \operatorname{Re} p > 0 \tag{3.17}$$

where M is an arbitrary constant.

Analogous estimates for the contour L_3 show that $\sigma_0^+(p) = O(p^{-3/2})$, $p \to \infty$ and, consequently

$$\sigma^+(p)/\sigma_0^+(p) = O(p), \quad p \to \infty; \quad \sigma^+(p) = (Mp + N)\sigma_0^+(p), \quad \operatorname{Re} p > 0$$

where M and N are arbitrary constants.

By virtue of (3.10)–(3.12), for both the cleavage and delamination problems when $q > 3$ and $p \to \infty$, we have

$$\sigma^+(p)/\sigma_0^+(p) = O(p^\gamma), \quad \gamma < 0$$

Hence, by Liouville's theorem, it follows from this that $\sigma^+(p) \equiv 0$ and, consequently, non-trivial solutions of the problems in question do not exist when $q > 3$.

The functions

$$G(p) = \sigma^+(p)/N_1(p), \quad \mathrm{Re}\,p>0; \quad G(p) = \upsilon^-(p)/N_2(p), \quad \mathrm{Re}\,p<0$$

and the corresponding solutions (2.5)–(2.12) are determined from the general solutions $\sigma^+(p)$ using formulae (3.1).

When $x < 0$, we can represent these solutions in the form of series over the residues in the zeros of the function $N_1(p)$. On closing the contours L_1 and L_2 on the right with semicircles of large radius, we obtain

$$\begin{aligned} u_k(x, y) &= u_{kq}(x, y) \equiv \frac{M}{2\pi i}\int\limits_{L_q} \frac{\overset{+}{\sigma}_{0q}(p)}{N_1(p)} U_k(p, y) e^{px} dp = \\ &= M\left\{\frac{\overset{+}{\sigma}_{0q}(0) r(y)}{\mu h(1-b^2)}\delta_{1k} + \frac{\overset{+}{\sigma}_{0q}(0)x + \overset{+*}{\sigma}_{0q}(0)}{\mu h(1-b^2)}\delta_{2k} - \sum_{n=1}^{\infty}\left[\frac{\overset{+}{\sigma}_{0q}(a_n)}{N_1'(a_n)} U_k(a_n, y) e^{a_n x} + \right.\right. \\ &\left. + \frac{\overset{+}{\sigma}_{0q}(\bar{a}_n)}{N_1'(\bar{a}_n)} U_k(\bar{a}_n, y) e^{\bar{a}_n x}\varepsilon_n\right] - \frac{\overset{+}{\sigma}_{0q}(2(-1)^{q+1} i\beta_0)}{N_1'(2(-1)^{q+1} i\beta_0)} U_k(2(-1)^{q+1} i\beta_0, y) \times \\ &\left. \times (\cos 2\beta_0 x + i(-1)^{q+1} \sin 2\beta_0 x)\right\} \end{aligned} \tag{3.18}$$

Here

$$\begin{aligned} r(y) &= -y + [4a^2 - (a^2+1)(b^2+1)]h\alpha^{-1} \\ \overset{+}{\sigma}_{02}(0) &= -\overset{+}{\sigma}_{01}(0) = i\mu\sqrt{2h\alpha_- a^{-1}}, \quad \overset{+*}{\sigma}_{02}(0) = -\overset{+*}{\sigma}_{01}(0) + \frac{\overset{+}{\sigma}_{01}(0)}{i\beta_0} \\ \overset{+}{\sigma}_{01}(2i\beta_0) &= 4i\beta_0 HQ(2i\beta_0)\sqrt{K_2(2i\beta_0)} \exp\left\{\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} \frac{\ln K_2(t)}{t - 2i\beta_0} dt\right\} \\ \overset{+}{\sigma}_{02}(-2i\beta_0) &= \overline{\overset{+}{\sigma}_{01}(2i\beta_0)}, \quad \alpha = 4a^2 - (1+b^2)^2 \end{aligned} \tag{3.19}$$

and δ_{qk} is the Kronecker delta. If a_n is a complex number, then $\varepsilon_n = 1$ and, if a_n is a real number, then $\varepsilon_n = 0$, $\mathrm{Re}\,a_n > 0$. An asterisk denotes a derivative with respect to p and a bar indicates a complex conjugate.

The first two terms on the right-hand side of equality (3.18), the residues at a zero, determine the polynomial solution (a continuous wave) in which the displacement $u(x, y)$ is bounded, $\upsilon(x, y)$ increases linearly along x and $\tau_{xy} = \sigma_x = \sigma_y = 0$. The residues at the points $p = \pm 2i\beta_0$ are continuous trigonometric elastic waves, and a series defines the waves, which decay exponentially with respect to x when $x \to -\infty$. In accordance with relations (3.15)–(3.17), the stress intensity factor is expressed, regardless of $q = 1$, 2, by the formula

$$K_I = -M\frac{2\sqrt{2}\mu\alpha_-}{a(1-b^2)} \tag{3.20}$$

$K_I > 0$ when $M < 0$ and $K_I \to K_I^0 = -2\sqrt{2}M\mu(\lambda+\mu)(\lambda+2\mu)^{-1}$ when $c \to 0$.

We will now consider the asymptotic form of the real waves when $x \to -\infty$. Adding and subtracting the solutions (3.18) with a factor of ½ when $q = 1, 2$, we obtain respectively

$$u_{k+}(x, y) = M\frac{\overset{+}{\sigma}_{01}(0)}{2i\beta_0\mu h(1-b^2)}\delta_{2k} - M\,\mathrm{Re}\,T_k(x, y) \tag{3.21}$$

$$u_{k-}(x, y) = \frac{M}{\mu h(1-b^2)}\{-i\sigma_{01}^{+}(0)r(y)\delta_{1k} + [-i\sigma_{01}^{+}(0)x + \mathrm{Im}\sigma_{01}^{+*}(0)]\delta_{2k}\} -$$
$$-M\mathrm{Im}T_k(x, y) \tag{3.22}$$
$$T_k(x, y) = \frac{\sigma_{01}^{+}(2i\beta_0)}{N_1'(2i\beta_0)}U_k(2i\beta_0, y)(\cos 2\beta_0 x + i\sin 2\beta_0 x)$$

In this case, the solution $u_{k+}(x, y)$ has the stress intensity factor (3.20) for $u_{k-}(x, y)$ $K_1 = 0$.

The first term on the right-hand side of equality (3.21) is a positive constant quantity as, according to (3.20), $K_I > 0$ for $M < 0$ in the cleavage problem, and the second term is proportional to the quantity $\cos(2\beta_0 x + \phi)$, where ϕ is a certain phase shift. It follows from this that, over certain ranges of x, the function $\upsilon(x, 0)$ can periodically have negative values and the crack surfaces then overlap. Hence, starting out from the requirements of mechanics, the asymptotic form of (3.21) and the corresponding complete solution of (3.18) must be checked for the condition $\upsilon(x, 0) > 0$.

The solution $u_{k-}(x, y)$, in which $K_I = 0$, refers to the delamination problem. For large x, due to the second term on the right-hand side of Eq. (3.22), the quantity $\upsilon(x, 0) > 0$ if $M < 0$ and the crack surfaces do not overlap. Additional checking of the solution is necessary in the case of arbitrary values of x.

We will now investigate the energy fluxes transported by the continuous waves and initially calculate the energy flux generated by the polynomial wave (the P-wave) (3.22) which is excited at $-\infty$. Since, in this wave,

$$\frac{\partial u^P}{\partial x} = 0, \quad \frac{\partial \upsilon^P}{\partial x} = -M\frac{i\sigma_{01}^{+}(0)}{\mu h(1-b^2)}, \quad \tau_{xy}^P = \sigma_x^P = \sigma_y^P = 0 \tag{3.23}$$

then, in the section $x = \text{const} < 0, y \in (-h, h)$, the flux is solely determined by the kinetic energy of the velocities of the displacements of the strip

$$E^P = \rho c\int_0^h\left[\left(\frac{\partial u^P}{\partial t}\right)^2 + \left(\frac{\partial \upsilon^P}{\partial t}\right)^2\right]dy$$

and, according to relations (3.23) and (3.19), we find

$$E^P = 2M^2\frac{\mu c\alpha_-}{a(1-b^2)} \tag{3.24}$$

From expression (3.2), we have

$$M = -K_I\frac{a(1-b^2)}{2\sqrt{2}\mu\alpha_-} \tag{3.25}$$

Substituting expression (3.25) into relation (3.24), we obtain an equality which relates the energy flux to the stress intensity factor

$$E^P = K_I^2\frac{ac(1-b^2)}{4\mu\alpha_-}$$

which is identical to the Kostrov–Nikitin–Flitman formula [7] for a flux from the crack tip during the cleavage of an elastic domain under steady-state conditions. Since the function $N_2(p)$ does not have pure imaginary zeros, there are no continuous waves when $x \to +\infty$, and this means that there is also no flux to $+\infty$. When account is taken of the fact that $K_I = 0$ in this case and, consequently, the overall flux into the crack tip during delamination, which is determined by the solution of (3.22), is equal to zero, it can be concluded, starting from the law of conservation of energy flux and from the conditions on the edges of the strip, that the flux E^P generated by the polynomial wave is compensated by the opposing energy flux E^T generated by the solitary trigonometric wave (the T-wave) (3.22) and the interaction energy E^{PT} of these waves. These fluxes can be found using the formula [8]

$$E = -2\int_0^h \left[\sigma_{i1}\frac{\partial u_i}{\partial t} + \frac{1}{2}\left(\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + \sigma_{ij}u_{i,j}\right)c\right]dy \tag{3.26}$$

where the first term in the square brackets (in tensor notation) determines the intensity of the external load applied to the end of the strip and moving in the x direction at a velocity $\partial u/\partial t$, and the second and third terms take account of the kinetic and internal strain energy respectively. Since, by virtue of relations (3.26), (3.23) and (3.19)

$$E^{PT} = -2\rho c\int_0^h \frac{\partial v^P}{\partial t}\frac{\partial v^T}{\partial t}dy = 2M\mu c\sqrt{\frac{2\alpha_-}{ah}}\int_0^h \frac{\partial v^T}{\partial x}dy$$

the fluxes E^P and E^{PT} only contain a kinetic component. It is practically impossible to specify this component without deforming the strip at $-\infty$. The mechanical treatment of the solutions therefore consists of the following. In the delamination problem, the stresses σ_x^T and σ_{xy}^T are specified at the ends of the strips. The energy flux E^T due to them is completely compensated by the counterflux of kinetic energy E^P in the strip, as a solid body, and the kinetic energy of the interaction of the waves E^{PT}. The load at the ends of the strips x = const, $y \in (0, h)$ leads to their gathering of momentum, which is proportional to x. The solution does not have an analogue in the quasi-steady-state case where the load at the ends of the strips causes displacements $v(x, y)$ in the form of a third degree polynomial in x.

An additional condition is required for determining the arbitrary constant M but the solution (3.22) which has been obtained cannot be independently implemented using just any M. It is obvious that the inequality $\sigma_y(x, 0) < 0$ for any $x > 0$, which is necessary in the delamination problem, is not satisfied in this solution. However, if it is specified that there is some compressive overloading of the strip, then M can be found from different mechanical conditions of the type $\sigma_y(x, 0) \le \sigma_c$ when $x > 0$, where σ_c is the minimum permissible compressive stress. Any stress $\sigma_y(x, 0)$ is permitted in the cleavage problem when $x > 0$ but, when expression (3.21) is used, the problem of the overlapping of the crack surfaces at $-\infty$ arises, which requires additional checking.

Knowing the value of K_{IC}, that is, the limiting stress intensity factor for a given material, it is possible to find M. From Irwin's fracture condition $K_I = K_{IC}$ and expression (3.20), we have

$$M = -K_{IC}\frac{a(1-b^2)}{2\sqrt{2}\mu\alpha_-} \tag{3.27}$$

When $c \to 0$, the quantity $M \to -K_{IC}(\lambda + 2\mu)/[2\sqrt{2}\mu(\lambda + \mu)]$. If the amplitude

$$-M \ge K_{IC}(\lambda + 2\mu)/[2\sqrt{2}\mu(\lambda + \mu)]$$

is given, the corresponding velocity can be found from formula (3.27) as an algebraic equation.

The T-waves (3.21) and (3.22) in the section $x = x_0 < 0$ can be represented in the form

$$u_{k+}^T(x, y) = -M\mathrm{Re}[(A_k + iB_k)(\cos 2\beta_0 x + i\sin 2\beta_0 x)] \tag{3.28}$$

$$u_{k-}^T(x, y) = -M\mathrm{Im}[(A_k + iB_k)(\cos 2\beta_0 x + i\sin 2\beta_0 x)] \tag{3.29}$$

where A_k and B_k are certain functions of β_0 and y. In the section $x = x_0 + \pi(4\beta_0)^{-1}$, the second wave has the form

$$u_{k-}^T = -M(A_k\cos 2\beta_0 x_0 - B_k\sin 2\beta_0 x_0)$$

and is identical with the first. Since the energy fluxes in the intervals $x \in (-\infty, 0)$ and $x \in (0, \infty)$ remain constant, the flux generated by the second wave in the section $x = x_0 + \pi(4\beta_0)^{-1}$ will be the same as when $x = x_0$ and, consequently, the energy fluxes generated by the T-waves (3.28) and (3.29) are identical.

We will now consider the skew-symmetric problem B. Taking account of the properties of the functions $N_3(p)$ and $N_4(p)$ and starting out from the earlier principles, we split Eq. (3.6) into the two Riemann problems

$$\tau_j^+(p) = K_j(p)u_j^-(p), \quad j = 1, 2, \quad p \in L \tag{3.30}$$

$$\tau^+(p) = \tau_1^+(p)\tau_2^+(p), \quad u^-(p) = u_1^-(p)u_2^-(p)$$

$$K_1(p) = H_1 p \operatorname{tg}\pi p \frac{\operatorname{tg}\pi(p-2i\beta_0)\operatorname{tg}\pi(p+2i\beta_0)}{\operatorname{tg}\pi(p-i\beta_0)\operatorname{tg}\pi(p+i\beta_0)}, \quad K_2(p) = \frac{K(p)}{K_1(p)}$$

$$K_1(p) \sim H_1\pi\frac{\operatorname{th}^2 2\pi\beta_0}{\operatorname{th}^2\pi\beta_0}p^2, \quad p\to 0; \quad K_1(i\beta)\sim -H_1|\beta|, \quad \beta\to\pm\infty; \quad H_1 = \frac{2\mu\alpha_-}{b(1-b^2)}$$

$$K_2(0) = \lim_{p\to 0} K_2(p) = \frac{\alpha b h \operatorname{th}^2\pi\beta_0}{2\pi\alpha_-\operatorname{th}^2 2\pi\beta_0} > 0; \quad \lim_{p\to\infty} K_2(p) = 1$$

$$\tau_{20}^+(p) = \exp\left\{-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln K_2(t)}{t-p}dt\right\}, \quad \operatorname{Re} p > 0; \quad \lim_{p\to\infty}\tau_{20}^+(p) = 1$$

$$\tau_{20}^+(i\beta) = \sqrt{K_2(i\beta)}\exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln K_2(t)}{t-i\beta}dt\right\}, \quad \tau_{20}^+(0) = \sqrt{K_2(0)} \tag{3.31}$$

Here, $\tau_{20}^+(p)$ is the canonical solution of the second Riemann problem (3.30). The function $K_2(i\beta)$ is strictly positive on the imaginary axis and, consequently, its index is equal to zero. The contours L_q, on passing along the imaginary axis, now go around to the left and to the right of the five points: $p = 0, p = \pm i\beta_0, p = \pm 2i\beta_0$. Correspondingly, solutions can be sought for 2^5 contours L_q ($q = 4, 5, \ldots$).

Factorizing the tangents on the contour L_4 (Fig. 2*a*), we obtain the canonical solution of the first Riemann problem

$$\tau_{14}^+(p) = H_1 Q_1(p)(p^2+4\beta_0^2)(p^2+\beta_0^2)^{-1}, \quad u_{14}^-(p) = [p^2 Q_1(-p)]^{-1}$$

$$Q_1(p) = Q(p)\frac{\Gamma(1+p-i\beta_0)\Gamma(1+p+i\beta_0)}{\Gamma(1/2+p-i\beta_0)\Gamma(1/2+p+i\beta_0)}$$

where

$$\tau_{14}^+(p)\sim H_1 p^{-1/2}, \quad p\to\infty; \quad \tau_{14}^+(0) = 2H_1\sqrt{\pi}\operatorname{th}2\pi\beta_0/\operatorname{th}\pi\beta_0 > 0 \tag{3.32}$$

Hence,

$$\tau_{04}^+(0) = \tau_{14}^+(0)\sqrt{K_2(0)} = \frac{2\mu}{1-b^2}\sqrt{\frac{2h\alpha\alpha_-}{b}}$$

Similarly on the contour L_5 (Fig. 2*b*)

$$\tau_{15}^+(p) = H_1 Q_1(p), \quad \tau_{15}^+(p)\sim H_1 p^{-1/2}, \quad p\to\infty, \quad \tau_{15}^+(0) = \frac{1}{4}\tau_{14}^+(0) \tag{3.33}$$

We now construct the general solution of Eq. (3.6) on the contours L_4 and L_5. It can be shown that, on other contours, either a solution does not exist or it is a linear combination of those already obtained. On the contours L_4 and L_5, we have

$$\frac{\tau^+(p)}{\tau_{0q}^+(p)} = \frac{u^-(p)}{u_{0q}^-(p)}, \quad \tau_{0q}^+(p) = \tau_{1q}^+(p)\tau_{20}^+(p), \quad u_{0q}^-(p) = u_{1q}^-(p)u_{20}^-(p) \tag{3.34}$$

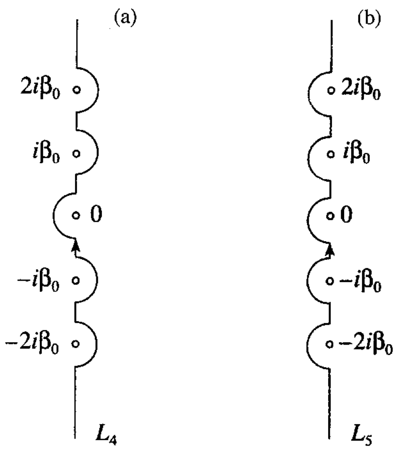

Fig. 2

Since the solution of the initial problem, which has a bounded energy in the neighbourhood of the crack surfaces, is representable in the form $\tau_{xy}(x, 0) \sim K_{II}/\sqrt{2\pi x}, x \to 0$, then $\tau^+(p) \sim K_{II}/\sqrt{2p}, p \to 0$ and, according to relations (3.31)–(3.33), the ratios (3.34) are constant when $p \to \infty$. Using the generalized Liouville theorem, we obtain from this

$$\tau^+(p) = M_q \overset{+}{\tau}_{0q}(p), \quad K_{II} = M_q \frac{2\sqrt{2}\mu\alpha_-}{b(1-b^2)}, \quad q = 4, 5 \tag{3.35}$$

where M_q are arbitrary constants and K_{II} is the stress intensity factor which, apart from terms up to M_q, is independent of q. As in the case of a normal cleavage crack, if the velocity c and the fracture characteristics of the material K_{IIC} are specified, the equality

$$M_q = K_{IIC} \frac{b(1-b^2)}{2\sqrt{2}\mu\alpha_-} \tag{3.36}$$

serves for calculating M_q and, if K_{IIC} and M_q are given, c can be found from Eq. (3.6). Since

$$K_{II}^0 = \lim_{c \to 0} K_{II} = M_q \frac{2\sqrt{2}\mu(\lambda+\mu)}{\lambda+2\mu}$$

then the infinitesimal load amplitude (in the case of the infinitesimal velocity $c \to 0$) is determined by the formula

$$M_q = K_{IIC} \frac{\lambda+2\mu}{2\sqrt{2}\mu(\lambda+\mu)}$$

We will now consider a problem associated with the propagation of waves and with energy fluxes. On expanding solution (2.5) in series in the residues to the right of L_4 when $x < 0$, we obtain

$$\begin{aligned} u_{k4}(x, y) &= \frac{M_4}{2\pi i}\int_{L_4} \frac{\overset{+}{\tau}_{04}(p)}{N_3(p)} U_k(p, y) e^{px} dp = -M_4 u_k^P(x, y) - \\ &- M_4 \sum_{n=1}^{\infty} \left[\frac{\overset{+}{\tau}_{04}(a_n)}{N_3'(a_n)} U_k(a_n, y) e^{a_n x} + \frac{\overset{+}{\tau}_{04}(\bar{a}_n)}{N_3'(\bar{a}_n)} U_k(\bar{a}_n, y) e^{\bar{a}_n x} \varepsilon_n \right] \end{aligned} \tag{3.37}$$

where

$$u_1^P(x,y) = \frac{1-b^2}{\alpha\mu h}[\overset{+}{\tau}_{0q}(0)x + \overset{+*}{\tau}_{0q}(0)],\quad u_2^P(x,y) = s(y)\overset{+}{\tau}_{0q}(0),\quad u_3^P(x,y) = 0$$

$$u_4^P(x,y) = 4(a^2-b^2)(\alpha h)^{-1}\overset{+}{\tau}_{0q}(0),\quad u_5^P(x,y) = 0,\quad q = 4 \tag{3.38}$$

$$s(y) = \{a^2 + (1-2a^2+b^2)[yh^{-1} - (1-b^2)^{-1}]\}(\alpha\mu)^{-1}$$

Substituting expressions (3.38) into formula (3.36), we find the intensity of the energy flux generated by the solitary P-wave which is excited at $-\infty$,

$$E_{4-}^P = 4K_{II4}^2 R,\quad R = \frac{bc(1-b^2)}{4\mu\alpha_-}$$

Since it exceeds the flux intensity in the Kostrov–Nikitin–Flitman formula, corresponding to the given K_{II} and to flux from the crack tip by a factor of four, it follows from the law of conservation of flux that 3/4 of this intensity, on by-passing the crack, departs to $+\infty$ with the T-wave, which is determined by the residues at the points $p = \pm i\beta_0$

$$u_{k4+}^T(x,y) = 2M_4\mathrm{Re}\left[\frac{\bar{u}_{04}(i\beta_0)}{N_4'(i\beta_0)}U_k(i\beta_0,y)(\cos\beta_0 x + i\sin\beta_0 x)\right],\quad k = 1,2$$

$$E_{4+}^T = 3K_{II}^2 R \tag{3.39}$$

The solution on the contour L_5 when $x \to -\infty$ comprises both P-waves (3.38), where $q = 5$,

$$\overset{+}{\tau}_{05}(0) = \frac{1}{4}\overset{+}{\tau}_{04}(0),\quad \overset{+*}{\tau}_{05}(0) = \frac{1}{4}\overset{+*}{\tau}_{04}(0)$$

and T-waves

$$u_{k5-}^T(x,y) = -2M_5\mathrm{Re}\left[\frac{\overset{+}{\tau}_{05}(2i\beta_0)}{N_3'(2i\beta_0)}U_k(2i\beta_0,y)(\cos 2\beta_0 x + i\sin 2\beta_0 x)\right] \tag{3.40}$$

When $x \to +\infty$, it is obvious that there are no propagating waves.

Forming the linear combinations

$$u(x,y) = u_{15}(x,y) - \frac{1}{4}u_{14}(x,y),\quad v(x,y) = u_{25}(x,y) - \frac{1}{4}u_{24}(x,y) \tag{3.41}$$

where $M_4 = M_5$ we obtain a solution which does not contain P-waves. Only the T-wave (3.40) with a frequency $2\beta_0$ is excited in it at $-\infty$. This T-wave transports an energy flux with an intensity E_-^T and, beyond the crack tip, it is transformed into a T-wave with a frequency β_0

$$u_{k+}^T(x,y) = -\frac{1}{2}M_4\mathrm{Re}\left[\frac{\bar{u}_{04}(i\beta_0)}{N_4'(i\beta_0)}U_k(i\beta_0,y)(\cos\beta_0 x + i\sin\beta_0 x)\right]$$

According to formula (3.39), the flux intensity is

$$E_+^T = -\frac{3}{4}K_{II4}^2 R$$

In this solution, the stress intensity factor is expressed by the relation

$$K_{II} = K_{II5} - \frac{1}{4}K_{II4} = \frac{3}{4}K_{II4} = M_4\frac{3\mu\alpha_-}{\sqrt{2}b(1-b^2)} \tag{3.42}$$

and, by virtue of the Kostrov–Nikitin–Flitman formula, determines the intensity of the energy flux into the tip

$$E_0 = \frac{9}{16}K_{IIC}^2 R$$

Since $E_-^T = E_0 + E_+^T$, we have

$$E_-^T = -\frac{3}{16}K_{IIC}^2 R$$

This last formula shows that, in the solution being considered, a flux with an intensity $E_+^T < 0$, which is excited when $x \to \infty$, is directed towards the crack tip. It gives up part of its energy $E_0 > 0$ in fracturing the material and to the steady motion of the crack, and the remaining energy E_-^T departs to $-\infty$.

It is convenient to make use of the solutions (3.37) and (3.41) with the solitary P- and T-waves at $-\infty$ in constructing a Green's function.

For a given velocity c and $K_{II4} = K_{IIC}$, formula (3.42) determines the amplitude M_4 of the wave at $-\infty$ and, when M_4 and K_{IIC} are specified, the crack propagation velocity.

The investigation which has been carried out above naturally leads to the question of whether it is necessary that the energy flux from $-\infty$ in quasi-steady problems has to be directed as a whole from the strip across the crack tip. Strictly speaking, it is only possible to justify this hypothesis by solving the corresponding boundary-value problem in each case, proving, after investigating the piecewise-homogeneous solutions in the waveguide, that there are no continuous static waves in it when $x \to +\infty$.

4. CONCLUSION

We will summarise the basic results obtained in this paper. Solutions of homogeneous problems have been constructed taking account of the fact that the energy flux E^P from $-\infty$ generated by a solitary polynomial wave (solution (3.37), (3.38)), is much greater than the limiting magnitude E_C, since a large part of the energy, on by-passing the crack, departs with a trigonometrical wave along the positive semiaxis to $+\infty$. When there are propagating trigonometrical and polynomial waves, excited at $-\infty$, in the solution, the component of the energy flux which is generated by the polynomial wave is related to the stress intensity factor by a relation which is identical to the Kostrov–Nikitin–Flitman formula.

It has been shown that elastostatic solutions cannot be obtained from the steady solutions of the dynamic problem which have been considered by taking the limit when $c \to 0$.

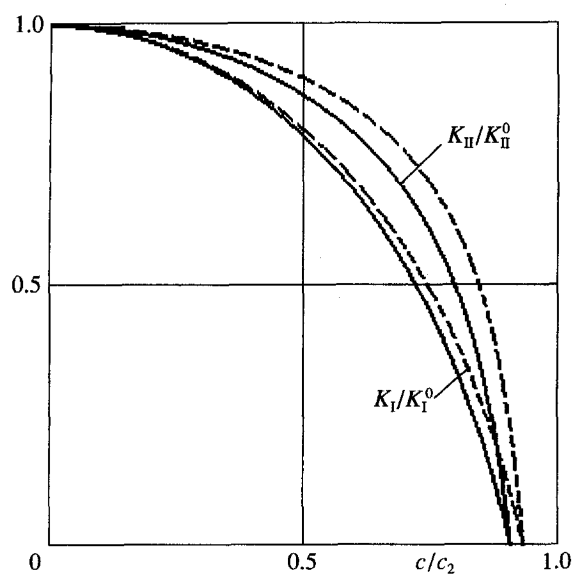

Fig. 3

The dynamic stress intensity factors K_I and K_{II} have been found at the tip of a crack which propagates along the axis of symmetry of a strip at a constant velocity. The dependence of the normalized quantities K_I/K_I^0, K_{II}/K_{II}^0, (K_I and K_{II} are determined by formulae (3.20) and (3.35)) on the normalized crack propagation velocity c/c_2 is shown in Fig. 3. Linearly elastic materials which simulate concrete ($E = 3.6 \times 10^4$ MPa, $\nu = 0.2$ and $\rho = 2.4$ t/m^3) and rocky soil ($E = 5 \times 10^3$ MPa, $\nu = 0.35$ and $\rho = 2.5$ t/m^3) were considered. The results of the calculations for the above-mentioned materials are represented by the solid and dashed curves respectively. As the velocity increases from zero up to the velocity of Rayleigh waves, the quantities being considered decrease monotonically, tending to zero, which is indicative of the possibility of the branching of the crack.

The fracture criteria (3.27) and (3.36), which relate the critical stress intensity factor to the crack propagation velocity and the amplitude of the propagating waves, have been obtained.

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